Model Categories by Example

Lecture 4

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Recap

Introduced cofibrant generation $\Rightarrow \exists$ suitable sets I, J such that 1) $LLP(RLP(I)) =$ cofibrations + smallness condition on codomains of I, J.

2) $LLP(RLP(J)) =$ acyclic cof.

sSet Kan, Top Quillen, ... etc cof. gen Top Strøm not cof. gen

Lack of fibrantly generated things $\rightsquigarrow$ cosmallness problems.

$$\operatorname{colim}_\beta \operatorname{Hom}(A, X_\beta) \cong \operatorname{Hom}(A, \operatorname{colim} X_\beta)$$

Recap

Right transferred model structures $U : \mathcal{D} \leftrightarrows \mathcal{E} : F$ model cat.

weak equiv + fib. detected with U.

* Special case projective model on $\mathcal{E}^D$ objectwise weak equiv + fib.

Existed if $\mathcal{E}$ cg. gen cofib gen locally presentable

* Left transfer. Harder! But special $\mathcal{E}$ was combinatorial we have injective model $\mathcal{E}^D$ objectwise weak equiv + cof.

$$\mathrm{id} : \mathcal{E}^D_{inj} \leftrightarrows \mathcal{E}^D_{proj} : \mathrm{id} \quad \text{Quillen Equivalence.}$$

Reedy model structures

A *Reedy category* is a category R equipped with two wide subcategories R_+ and R_- and a *degree function* $d\colon \mathrm{ob}(R) \to \mathbb{N}$ such that the following hold:

n.b no automorphisms on objects of R.

(1) every nonidentity morphism in R_+ raises degree;

(2) every nonidentity morphism in R_- lowers degree;

(3) every morphism in R factors uniquely as a morphism in R_- followed by a morphism in R_+.

<u>Ex</u> + Δ is Reedy

$[k] \to [n]$ is in Δ_+ if inj.

$[n] \to [k]$ is in Δ_- if surjective

* If R is Reedy, then R^{op} is Reedy

* Diagram shapes $(\bullet \to \bullet \to \cdots)$ $(\bullet \leftarrow \bullet \to \bullet)$

homotopy lim + colim.

Reedy model structures

Proposition: Let R be Reedy, and $\mathcal{E}$ <u>any</u> model structure. Then there is a model structure on $\mathcal{E}^R$ where a map is:

* weak equiv are objectwise.
* fib ⇝ "Reedy fibrations"
* cof ⇝ "Reedy cofibrations"

Explicit descriptions

Lemma:

$$\mathcal{E}^D_{inj} \underset{\longrightarrow}{\xleftarrow{id}} \mathcal{E}^D_{Reedy} \underset{\longrightarrow}{\xleftarrow{id}} \mathcal{E}^D_{proj}$$

all Quillen equiv.

Bousfield localization

Another method of "new from old"

Underlying category remains the same.

* increase weak Equivalences *

If we add more weak Equivs ⤳ acyclic cofibrations also increase

⇒ fibrations must decrease.

fix either the fib or cofibrations and RLP/LLP the other.

Homotopy function complexes

Idea: Model categories have hom-spaces between objects.

Let $\mathcal{C}$ be a model category, and $X \in \mathcal{C}$.

- A *cosimplicial resolution of X* is an acyclic cofibration $A^* \xrightarrow{\sim} \underbrace{cX^*}_{\text{constant cosimp. object at } X}$ in $\mathcal{C}^{\Delta}_{\text{Reedy}}$.
- A *simplicial resolution of X* is an acyclic fibration $\underbrace{cX_*}_{\text{constant simp. obj at } X} \xrightarrow{\sim} A_*$ in $\mathcal{C}^{\Delta^{op}}_{\text{Reedy}}$.

$\Rightarrow r: \mathcal{C} \to \mathcal{C}^{\Delta}$ $\quad \tilde{r}: \mathcal{C} \to \mathcal{C}^{\Delta^{op}}$ functorial (co)simplicial resolution functor.

Homotopy function complexes

$X, Y \in \mathcal{C}$ form a bisimplicial set $\mathrm{Hom}(rX, \tilde{r}Y)$

$$\mathrm{Hom}(rX, \tilde{r}X)_{n,k} := \mathrm{Hom}(rX^k, \tilde{r}Y_n)$$

* $\mathrm{map}(X, Y) := \mathrm{diag}\,(\mathrm{Hom}(rX, \tilde{r}Y)) \in sSet$.

"mapping space"

"homotopy function complex"

Homotopy function complexes

Prop: ① $\mathrm{Map}(X,Y) \in sSet$ is a Kan complex

$\Rightarrow$ fibrant in $sSet_{Kan}$

$\Rightarrow$ a space.

② Independent of any choices.

③ If $\mathcal{E}$ is a simplical model category.

$$\mathrm{map}(X,Y) \simeq \underline{\mathrm{hom}}(X,Y)$$

cof fib

4 $\pi_0 \mathrm{map}(X,Y) = [X,Y]$

Left Bousfield localization

Let $\mathcal{C}$ be a model category and S be a set of morphisms in $\mathcal{C}$.

- An object $Z \in \mathcal{C}$ is said to be *S-local* if

$$\operatorname{map}(s, Z) \colon \operatorname{map}(B, Z) \to \operatorname{map}(A, Z)$$

  is a weak equivalence in $\mathbf{sSet}_{\mathrm{Kan}}$ for all $s \colon A \to B$ in S.

- A morphism $f \colon X \to Y$ in $\mathcal{C}$ is an *S-equivalence* if

$$\operatorname{map}(f, Z) \colon \operatorname{map}(Y, Z) \to \operatorname{map}(X, Z)$$

  is a weak equivalence in $\mathbf{sSet}_{\mathrm{Kan}}$ for all S-local objects $Z \in \mathcal{C}$.

Left Bousfield localization

A *left Bousfield localization* of a model category $\mathcal{C}$ with respect to a set of morphisms S (if it exists) is a new model structure $L_S\mathcal{C}$ on the underlying category of $\mathcal{C}$ such that the:

- weak equivalences are the S-local equivalence (S-local equiv $\supset$ weak equiv)
- fibrations RLP(acyclic cof.)
- cofibrations are the cof. of $\mathcal{C}$

The f.b. obj of $L_S\mathcal{C}$ are the S-local f.b. obj of $\mathcal{C}$.

Left Bousfield localization

Lemma: $id : \mathcal{C} \to L_S \mathcal{C}$ is left Quillen.

Proposition: The S-local equivs between S-local fib obj are exactly the original weak equiv in $\mathcal{C}$.

Proposition: " " fibrations " " fibrations

An aside

<u>Prop</u> A pushout of (acyclic) cofib is an (acyclic) cofibration.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r] & Y' \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

f' has LLP w.r.t any map that f has LLP w.r.t.

Dually for fibrations

Non-existence of localizations

$$\begin{tikzcd} a \arrow[r, two heads, "\sim"] \arrow[d, hook, "\wr"'] & b \arrow[d, hook, "\wr"] \\ c \arrow[r, hook, two heads] & d \end{tikzcd}$$

Proposition: The following is a model category:

want to localise at the map $S = \{a \to c\}$

* The S-local objects are c, d "(same maps from a and c)";

* The map $C \to D$ was <u>not</u> a weak equivalence before so it is still not a weak equiv.

* If $a \overset{\sim}{\hookrightarrow} c \Rightarrow b \overset{\sim}{\hookrightarrow} d$ as a pushout of an acyclic cof"

By 2-out-of-3 $\Rightarrow c \xrightarrow{\sim} d$ ↯

Left properness

A model category $\mathcal{C}$ is *left proper* if for every weak equivalence $a \xrightarrow{\sim} b$, and cofibration $a \hookrightarrow c$, the morphism f in the following pushout is also a weak equivalence:

$$\begin{tikzcd} a \arrow[r, "\sim"] \arrow[d, hook] & b \arrow[d] \\ c \arrow[r, "f"'] & p \end{tikzcd} .$$

Proposition: If all obj are cofibrant $\Rightarrow$ left proper.

sSet$_{\text{Kan}}$

Top$_{\text{Quillen}}$

Existence of localizations

Proposition: If $\mathcal{C}$ is combinatorial + left proper, then left Bous. loc^n of $\mathcal{C}$ at any set S exists!

"But what about Top called ?"

Cellular model categories

$$\begin{tikzcd} & \text{Cofibrant gen} & \\ \text{Combinatorial} \arrow[ur, Rightarrow] \arrow[rr, phantom, "\sim"] & & \text{Cellular} \arrow[ul, Rightarrow] \end{tikzcd}$$

"Cellular $\Rightarrow$ Compactness + smallness condition on I, J
cofibrations are effective monomorphisms"

$sSet_{Kan}$ is Cell + Comb

$Top_{Quillen}$ is Cell

$Set_{Trivial}$ is Comb but not Cellular

$* \sqcup * \to *$

Existence of localizations

Proposition: If $\mathscr{C}$ is cellular + left proper $\Rightarrow$ LBL exist for S a set.

Localizations at homology theories

Let h_* be a generalized coh. theory.

$\xRightarrow{\text{Bousfield}}$ Can form a localization of Top Quillen at those maps $f: X \to Y$ such that $h_* f: h_* X \xrightarrow{\cong} h_* Y$

$\Rightarrow$ Bousfield also proves this for spectra.

Postnikov towers

* Again work with $\mathrm{Top}_{\text{Quillen}}$ $\quad S_n: \{S^k \to * \mid k > n\}$

$$L_{S_n} \mathrm{Top}_{\text{Quillen}} = \mathrm{Top}_{\leq n}.$$

$\rightsquigarrow$ weak equivs are those map $X \to Y$ such that $\pi_i(X) \to \pi_i(Y)$ isos for $i \leq n$

$$\cdots \longrightarrow \mathrm{Top}_{\leq 1} \xrightarrow{\mathrm{id}} \mathrm{Top}_{\leq 0}$$

Completions

Let $Ch(\mathbb{Z})_{proj}$

$W =$ Quasi-isos $\qquad \wedge = \otimes$

$F =$ degreewise epis.

Form $\mathbb{Z}/p$-homology isos $\Rightarrow$ $f: X \to Y$ is such a map if $f \wedge \mathbb{Z}/p$ is a weak equiv.

$L_{\mathbb{Z}/p}\, Ch(\mathbb{Z})_{proj} \sim$ "p-complete abelian grps" $\sim D(\mathbb{Z})^{\wedge}_p$

Right Bousfield localization

Let $\mathcal{C}$ be a model category and K a set of objects of $\mathcal{C}$.

- A morphism $f\colon A \to B$ in $\mathcal{C}$ is a *K-coequivalence* if

  $$\mathrm{map}(X, f)\colon \mathrm{map}(X, A) \to \mathrm{map}(X, B)$$

  is a weak equivalence in $\mathbf{sSet}_{\mathrm{Kan}}$ for each $X \in K$.
- An object $Z \in \mathcal{C}$ is *K-colocal* if

  $$\mathrm{map}(Z, f)\colon \mathrm{map}(Z, A) \to \mathrm{map}(Z, B)$$

  is a weak equivalence in $\mathbf{sSet}_{\mathrm{Kan}}$ for any K-coequivalence.

Right Bousfield localization

The *right Bousfield localization* at K of $\mathcal{C}$ is the model category $R_K\mathcal{C}$ with underlying category of $\mathcal{C}$ such that the:

- weak equivalences K-colocal equivs
- fibrations are fib's of $\mathcal{C}$
- cofibrations LLP(Acyclic fib).

$\mathcal{C} \to R_K \mathcal{C}$ is right Quillen.

Right Bousfield localization

Proposition: Let $\mathcal{C}$ be right proper + either cellular or combinatorial then RBL exists for any set of objects.